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Series and ε-expansion of the hypergeometric functions

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Recent progress in analytical calculation of the *multiple* {*inverse, binomial, harmonic*} *sums* , related with ε-expansion of the hypergeometric function of one variable are discussed.

1. In the framework of the dimensional regularization [1] many Feynman diagrams can be written as hypergeometric series of several variables [2] (some of them can be equal to the rational numbers). This result can be deduced via Mellin-Barnes technique [3] or as solution of the differential equation for Feynman amplitude [4]. However, for application to the calculation of radiative corrections mainly the construction of the ε-expansion is interesting. At the present moment, several algorithms for the ε-expansion of different hypergeometric functions have been elaborated. They were mainly related to calculations of concrete Feynman diagrams [5]. Only recently, the general algorithm for integer values of parameters has been described in [6] and its generalization has been done in [7]. The results of expansion are expressible in terms of the new functions, like harmonic polylogarithms [8] or their recent generalization [6,9]. Let us shortly describe, how this algorithm woks on the example of the generalized hypergeometric function of one variable. The starting point is the series representation:

$$
{}_PF_Q\left(\begin{array}{c}\{A_1+a_1\varepsilon\},\{A_2+a_2\varepsilon\},\cdots\{A_P+a_P\varepsilon\}\\ \{B_1+b_1\varepsilon\},\{B_2+b_2\varepsilon\},\cdots\{B_Q+b_q\varepsilon\}\end{array}\middle| z\right)
= \sum_{j=0}^{\infty} \frac{z^j}{j!}\,\frac{\Pi_{s=1}^{P}(A_s+a_s\varepsilon)_j}{\Pi_{r=1}^{Q}(B_r+b_r\varepsilon)_j} ,
$$

where $(\alpha)_j \equiv \Gamma(\alpha+j)/\Gamma(\alpha)$ is the Pochhammer symbol. We concentrate on the case ${}_{Q+1}F_Q$, when series converges for all $|z|<1$, and on the integer or half-integer values of the parameters $\{A_i,\ B_j\} \in \{m_i, m_j+\frac{1}{2}\}$. To perform the ε-expansion we use the well-known representation

$$
\frac{(m+a\varepsilon)_j}{(m)_j} =
\exp\left\{-\sum_{k=1}^{\infty}\frac{(-a\varepsilon)^k}{k}\left[S_k(m+j-1)-S_k(m-1)\right]\right\} ,
$$

where m is an integer positive number, $m>1$ and $S_k(j)=\sum_{l=1}^{j} l^{-k}$ is the harmonic sum satisfying the relation $S_k(j)=S_k(j-1)+1/j^k$. For half-integer positive values $\mathcal{A}_i \equiv m_i+1/2>0$, we use the duplication formula

$$
\left(m+\frac{1}{2}+a\varepsilon\right)_j = \frac{(2m+1+2a\varepsilon)_{2j}}{4^j\,(m+1+a\varepsilon)_j} .
$$

To work only with positive values for parameters of hypergeometric function we can apply several times the Kummer relation:

$$
{}_PF_Q\left(\begin{array}{c}a_1,\cdots,a_P\\ b_1,\cdots,b_Q\end{array}\middle| z\right)=1
+z\frac{a_1\ldots a_P}{b_1\ldots b_Q}\;{}_{P+1}F_{Q+1}\left(\begin{array}{c}1,1+a_1,\cdots,1+a_P\\ 2,1+b_1,\cdots,1+b_Q\end{array}\middle| z\right) .
$$

After applying this procedure the original hypergeometric function can be written as

$$
{}_{P+1}F_P\left(\begin{array}{c}\{m_i+a_i\varepsilon\}^J,\{p_j+\frac{1}{2}+d_j\varepsilon\}^{P+1-J}\\ \{n_i+b_i\varepsilon\}^K,\{l_j+\frac{1}{2}+c_j\varepsilon\}^{P-K}\end{array}\middle| z\right)=
\sum_{j=1}^{\infty}\frac{z^j}{j!}\frac{1}{4^{j(K-J+1)}}\frac{\Pi_{i=1}^{J}(m_i)_j}{\Pi_{l=1}^{K}(n_l)_j}
\times\Pi_{r=1}^{P+1-J}\frac{(2p_r+1)_{2j}}{(p_r+1)_j}\Pi_{s=1}^{P-K}\frac{(l_s+1)_j}{(2l_s+1)_{2j}}\Delta , \qquad (1)
$$

with

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$$\Delta = \exp\Biggl[\sum_{k=1}^{\infty}\frac{(-\varepsilon)^k}{k}\Biggl(\sum_{\omega=1}^{K} b_\omega^k S_k(n_\omega+j-1)$$
$$-\sum_{i=1}^{J} a_i^k S_k(m_i+j-1)$$
$$+\sum_{s=1}^{P-K} c_s^k \left[S_k(2l_s+2j)-S_k(l_s+j)\right]$$
$$-\sum_{r=1}^{P+1-J} d_r^k \left[S_k(2p_r+2j)-S_k(p_r+j)\right]\Biggr)\Biggr] .$$

In this way, the ε-expansion of the hypergeometric function (1) is reduced to the calculation of the *multiple sums*

$$\sum_{j=1}^{\infty}\frac{z^j}{j!}\frac{1}{4^{j(K-J+1)}}\Pi_{i,r,\omega,s}\frac{(m_i-1+j)!(2p_r+2j)!}{(n_\omega-1+j)!(2l_s+2j)!}$$
$$\times[S_{a_1}(m_1+j-1)]^{i_1}\dots[S_{a_\mu}(m_\mu+j-1)]^{i_p}$$
$$\times[S_{b_1}(2p_r+2j)]^{j_1}\dots[S_{b_\nu}(2p_\nu+2j)]^{j_q},$$

where $\{m_j,n_k,l_\omega,p_r\}$ - positive integer numbers. In the calculation of massive Feynman diagrams [10–12] we get *multiple sums* of the following form,

$$\Sigma_{a_1,\dots,a_p;\,b_1,\dots,b_q;c}(u) \equiv \sum_{j=1}^{\infty}\frac{1}{\binom{2j}{j}^k}\frac{u^j}{j^c}$$
$$\times S_{a_1}\dots S_{a_p}\bar{S}_{b_1}\dots\bar{S}_{b_q}\ , \qquad (2)$$

where u is an arbitrary argument and we accept that the notation S_a and $\bar{S}_b$ will always mean $S_a(j-1)$ and $S_b(2j-1)$, respectively, even we do not mention this explicitly. When there are no sums of the type S_a or $\bar{S}_b$ in the r.h.s. of Eq. (2), we put a "$-$" sign instead of the indices (a) or (b) of Σ, respectively. Some indices (a) or (b) may be equal to each other, which is equivalent to power of a proper harmonic sum. For particular values of k, the sums (2) are called

$$k = \left\{\begin{array}{ll} 0 & \textit{harmonic} \\ 1 & \textit{inverse binomial} \\ -1 & \textit{binomial} \end{array}\right\} \text{ sums}$$

These sums are related to ε-expansion of the hypergeometric functions of type (1) with the following set of parameters:

$$m_i \in \{1\}^K,\ \{2\}^L\ , n_i \in \{1\}^R,\ \{2\}^{K+L-R-1-k}\ ,$$
$$p_j \in \{1\}^{J-k}\ , \quad l_j \in \{1\}^J, \quad u = 4^k z\ . \qquad (3)$$

In the recent paper [12], the sums of type (2) up to weight **4** have been studied in detail.

2. Let us rewrite the *multiple sums* (2) in the following form, $\Sigma^{(k)}_{A;B;c}(u) = \sum_{j=1}^{\infty} u^j \eta^{(k)}_{A;B;c}(j)$, where $A \equiv (a_1,\dots,a_p)$ and $B \equiv (b_1,\dots,b_q)$ denote the collective sets of indices, whereas $\eta^{(k)}_{A;B;c}(j)$ is the coefficient of u^j

$$\eta^{(k)}_{A;B;c}(j) = \frac{1}{\binom{2j}{j}^k}\frac{1}{j^c}S_{a_1}\dots S_{a_p}\bar{S}_{b_1}\dots\bar{S}_{b_q}\ . \qquad (4)$$

The idea is to find a recurrence relation with respect to j, for the coefficients $\eta^{(k)}_{A;B;c}(j)$ and then transform it into a differential equation for the *generating* function $\Sigma^{(k)}_{A;B;c}(u)$. In this way, the problem of summing the series would be reduced to solving a differential equation [13]. Using the explicit form of $\eta^{(k)}_{A;B;c}(j)$ given in Eq. (4), the recurrence relation can be written in the following form:

$$\left[2(2j+1)\right]^k (j+1)^{c-k}\eta^{(k)}_{A;B;c}(j+1)$$
$$= j^c \eta^{(k)}_{A;B;c}(j) + r^{(k)}_{A;B}(j)\ , \qquad (5)$$

where the explicit form of the "remainder" $r^{(k)}_{A;B}(j)$ is given by

$$\binom{2j}{j}^k r^{(k)}_{A;B}(j) = \prod_{k=1}^{p}\prod_{l=1}^{q}\Biggl\{\left[S_{a_k}+j^{-a_k}\right]^{i_k}\times$$
$$\left[\bar{S}_{b_l}+(2j)^{-b_l}+(2j+1)^{-b_l}\right]^{j_l}-\left[S_{a_k}\right]^{i_k}\left[\bar{S}_{b_l}\right]^{j_l}\Biggr\} . \qquad (6)$$

In other words, it contains all contributions generated by j^{-a_k}, $(2j)^{-b_l}$ and $(2j+1)^{-b_l}$ which appear because of the shift of the index j. Multiplying both sides of Eq. (5) by u^j, summing from 1 to infinity, and using the fact that any extra power of j corresponds to the derivative $u(\mathrm{d}/\mathrm{d}u)$, we arrive at the following differential equation for the generating function $\Sigma^{(k)}_{A;B;c}(u)$:

$$\left[\left(\tfrac{4}{u}-1\right)u\tfrac{\mathrm{d}}{\mathrm{d}u}-\tfrac{2}{u}\right]\left(u\tfrac{\mathrm{d}}{\mathrm{d}u}\right)^{c-1}\Sigma^{(1)}_{A;B;c}(u)$$
$$= \delta_{p0} + R^{(1)}_{A;B}(u)\ , \qquad (7)$$

$$\left(\tfrac{1}{u}-1\right)\left(u\tfrac{\mathrm{d}}{\mathrm{d}u}\right)^{c}\Sigma^{(0)}_{A;B;c}(u) = \delta_{p0}+R^{(0)}_{A;B}(u)\ , \qquad (8)$$

$$\left[\left(\tfrac{1}{u}-4\right)u\tfrac{\mathrm{d}}{\mathrm{d}u}-2\right]\left(u\tfrac{\mathrm{d}}{\mathrm{d}u}\right)^{c}\Sigma^{(-1)}_{A;B;c}(u)$$
$$= 2\delta_{p0} + 2\left(2u\tfrac{\mathrm{d}}{\mathrm{d}u}+1\right)R^{(-1)}_{A;B}(u)\ , \qquad (9)$$

where $R^{(k)}_{A;B}(u) \equiv \sum_{j=1}^{\infty} u^j r^{(k)}_{A;B}(j)$. The r.h.s. of

differential equation for sums includes the *multiple sums* with shifted index

$$G^{(k)}_{a_1,\ldots,a_p;\,b_1,\ldots,b_q;c} \equiv \sum_{j=1}^{\infty} \frac{1}{\binom{2j}{j}^k} \frac{u^j}{(2j+1)^c}$$

$$\times S_{a_1}\ldots S_{a_p}\bar{S}_{b_1}\ldots\bar{S}_{b_q} \equiv \sum_{j=1}^{\infty} u^j \nu^{(k)}_{A;B;c}(j) \;, \qquad (10)$$

where we accept the same conditions for the indices $\{a_i\}$ and $\{b_j\}$, as in the previous case. For investigation of these sums we again apply the generating function approach. In this case, the recurrence relations for the coefficients $\nu^{(k)}_{A;B;c}(j)$ are the following:

$$\left[2(2j+1)\right]^k (2j+3)^c \nu^{(k)}_{A;B;c}(j+1)$$
$$= (j+1)^k \left[(2j+1)^c \nu^{(k)}_{A;B;c}(j) + r^{(k)}_{A;B}(j)\right] \;, \quad (11)$$

with $r^{(k)}_{A;B}(j)$ given by Eq. (6). The proper set of differential equations is

$$\left[\left(\tfrac{4}{u}-1\right)u\tfrac{\mathrm{d}}{\mathrm{d}u} - \tfrac{2}{u} - 1\right]\left(2u\tfrac{\mathrm{d}}{\mathrm{d}u}+1\right)^c G^{(1)}_{A;B;c}(u)$$
$$= \delta_{p0} + \left(u\tfrac{\mathrm{d}}{\mathrm{d}u}+1\right) R^{(1)}_{A;B}(u) \;, \qquad (12)$$

$$\left(\tfrac{1}{u}-1\right)\left(2u\tfrac{\mathrm{d}}{\mathrm{d}u}+1\right)^c G^{(0)}_{A;B;c}(u) =$$
$$\delta_{p0} + \left(u\tfrac{\mathrm{d}}{\mathrm{d}u}+1\right) R^{(0)}_{A;B}(u) \;, \qquad (13)$$

$$\left[\left(\tfrac{1}{u}-4\right)u\tfrac{\mathrm{d}}{\mathrm{d}u} - 2\right]\left(2u\tfrac{\mathrm{d}}{\mathrm{d}u}+1\right)^c G^{(-1)}_{A;B;c}(u)$$
$$= 2\delta_{p0} + 2\left(2u\tfrac{\mathrm{d}}{\mathrm{d}u}+1\right) R^{(-1)}_{A;B}(u) \;. \qquad (14)$$

Equations, [(7), (12)], [(8), (13)], [(9), (14)], form the closed system of differential equations.

3. From the analysis, given in [12] we have deduced that the set of equations for *generating functions* has a simpler form in terms of a new variable. For *multiple inverse binomial sums* it is defined as

$$y = \frac{\sqrt{u-4}-\sqrt{u}}{\sqrt{u-4}+\sqrt{u}} \;, \quad u = -\frac{(1-y)^2}{y} \;, \qquad (15)$$

and for *multiple binomial sums* it has the following form:

$$\chi = \frac{1-\sqrt{1-4u}}{1+\sqrt{1-4u}}, \quad u = \frac{\chi}{(1+\chi)^2}. \qquad (16)$$

Let us consider the differential equation for *multiple inverse binomial sums* in terms of new variables. Equation (7) takes the form

$$\left(-\frac{1-y}{1+y}y\frac{d}{dy}\right)^{c-1} \Sigma^{(1)}_{A;B;c}(y) = \frac{1-y}{1+y}\sigma^{(1)}_{A;B}(y) \;, \quad (17)$$

where

$$y\frac{d}{dy}\sigma^{(1)}_{A;B} = \delta_{p0} + R^{(1)}_{A;B}(y) \;. \qquad (18)$$

Equation (17) could be rewritten as

$$\left(-\frac{1-y}{1+y}y\frac{d}{dy}\right)^{c-j} \Sigma^{(1)}_{A;B;c}(y) = \Sigma^{(1)}_{A;B;j}(y) \;, \qquad (19)$$

or, in equivalent form:

$$\left(-\frac{1-y}{1+y}y\frac{d}{dy}\right)^{c-j-1} \Sigma^{(1)}_{A;B;c}(y)$$
$$= \int_0^y dy \left(\frac{2}{1-y} - \frac{1}{y}\right) \Sigma^{(1)}_{A;B;j}(y) \;. \qquad (20)$$

From this representation we immediately get the following *statement*:

If for some j the series $\Sigma^{(1)}_{A;B;j}(y)$ are expressible in terms of *harmonic polylogarithms*, the sums $\Sigma^{(1)}_{A;B;j+i}(y)$ can also be presented in terms of *harmonic polylogarithms*. This follows from the definition of the *harmonic polylogarithms* (see Ref. [8]).

In a similar manner, let us rewrite the equation for generating function of the *multiple binomial sums*:

$$\left(\frac{1+\chi}{1-\chi}\chi\frac{\mathrm{d}}{\mathrm{d}\chi}\right)^{c} \Sigma^{(-1)}_{A;B;c}(\chi) = \frac{1+\chi}{1-\chi}\sigma^{(-1)}_{A;B}(\chi) \;, \quad (21)$$

$$\frac{1}{2}(1+\chi)^2\frac{\mathrm{d}}{\mathrm{d}\chi}\sigma^{(-1)}_{A;B}(\chi)$$
$$= \delta_{p0} + \left(2\frac{1+\chi}{1-\chi}\chi\frac{\mathrm{d}}{\mathrm{d}\chi}+1\right) R^{(-1)}_{A;B}(\chi) \;, \quad (22)$$

which could also be rewritten as

$$\left(\frac{1+\chi}{1-\chi}\chi\frac{\mathrm{d}}{\mathrm{d}\chi}\right)^{c-j} \Sigma^{(-1)}_{A;B;c}(\chi) = \Sigma^{(-1)}_{A;B;j}(\chi) \;, \qquad (23)$$

or, in an equivalent form:

$$\left(\frac{1+\chi}{1-\chi}\chi\frac{\mathrm{d}}{\mathrm{d}\chi}\right)^{c-j-1} \Sigma^{(-1)}_{A;B;c}(\chi)$$
$$= \int_0^\chi d\chi \left(\frac{1}{\chi} - \frac{2}{1+\chi}\right) \Sigma^{(1)}_{A;B;j}(\chi) \;. \qquad (24)$$

Again we get the previous statement.

4. The differential equation for *multiple inverse binomial sums* with the shifted index has a more complicated form. For their analysis let us use the geometrical variable [14] defined via $u_\theta \equiv 4\sin^2\frac{\theta}{2}$

$(0 \le u_\theta \le 4)$. In terms of this variable, Eq. (12) could be written as

$$\left[\cot\tfrac{\theta}{2}\tfrac{d}{d\theta} - \tfrac{1}{2\sin^2\frac{\theta}{2}} - 1\right]\left(2\tan\tfrac{\theta}{2}\tfrac{d}{d\theta}+1\right)^c G^{(1)}_{A;B;c}(u_\theta)$$

$$= \delta_{p0} + \left(1+\tan\tfrac{\theta}{2}\tfrac{d}{d\theta}\right) R^{(1)}_{A;B}(u_\theta) \, . \quad (25)$$

This equation can be decomposed into the system of differential equations

$$G^{(1)}_{A;B;c}(u_\theta) = \frac{1}{\sin\frac{\theta}{2}}\rho_{A;B;c}(\theta) \, , \quad (26)$$

$$\left(2\tan\tfrac{\theta}{2}\frac{d}{d\theta}\right)^c \rho_{A;B;c}(\theta) = \frac{\sin^2\frac{\theta}{2}}{\cos^3\frac{\theta}{2}} g_{A;B}(\theta) \, , \quad (27)$$

$$\tan\tfrac{\theta}{2}\frac{dg_{A;B}(\theta)}{d\theta} = \frac{d}{d\theta}\left(\sin^2\tfrac{\theta}{2}\, R^{(1)}_{A;B}(u_\theta) - \delta_{p0}\tfrac{1}{2}\cos\theta\right) \, . \quad (28)$$

The formal solution of Eq. (28) is

$$g_{A;B;c}(\theta) = \tfrac{1}{2}\sin\theta R^{(1)}_{A;B}(u_\theta) + \tfrac{1}{2}\int_0^\theta d\phi R^{(1)}_{A;B}(u_\phi) + \tfrac{1}{2}\delta_{p0}\left(\theta+\sin\theta\right) \, , \quad (29)$$

where $u_\phi = 4\sin^2\frac{\phi}{2}$. Substituting this result in Eq. (27) and integrating one time we get

$$\left(2\tan\tfrac{\theta}{2}\frac{d}{d\theta}\right)^{c-1} \rho_{A;B;c}(\theta) = -\sin\tfrac{\theta}{2} R^{(1)}_{A;B}(u_\theta) + \frac{1}{2\cos\frac{\theta}{2}}\int_0^\theta d\phi R^{(1)}_{A;B}(u_\phi) + \delta_{p0}\left(\frac{1}{2}\frac{\theta}{\cos\frac{\theta}{2}} - \sin\tfrac{\theta}{2}\right) + \int_0^\theta d\phi \sin\frac{\phi}{2}\frac{dR^{(1)}_{A;B}(u_\phi)}{d\phi} \, . \quad (30)$$

For $c=1$ the r.h.s. of Eq. (30) divided by $\sin\frac{\theta}{2}$ is the solution for the sum $G^{(1)}_{A;B;1}(u_\theta)$. Let us now apply this approach to the sum $G^{(1)}_{-;-;3}(u_\theta)$ which was not solved explicitly in Ref. [12]. Using the relation (see Eqs. (27))

$$\left(2\tan\tfrac{\theta}{2}\frac{d}{d\theta}\right)^{c-k} \rho_{A;B;c}(\theta) = \rho_{A;B;k}(\theta) \, , \quad (31)$$

and expression for $\rho_{-;-;2}(\theta)$ (see Eq. (2.74) in [12]), $\rho_{-;-;2}(\theta) = 2\mathrm{Ti}_2\left(\tan\frac{\theta}{4}\right) - \sin\frac{\theta}{2}$, we get after integration by parts

$$\rho_{-;-;3}(\theta) = \frac{1}{2}\int_0^\theta d\phi \cot\tfrac{\phi}{2}\rho_{-;-;2}(\phi)$$

$$= l_\theta\left[\rho_{-;-;2}(\theta) + \sin\tfrac{\theta}{2}\right] - \sin\tfrac{\theta}{2} - \tfrac{\theta}{4}\left[l_\theta^2 - L_\theta^2\right] - \frac{1}{2}\left[\mathrm{Ls}_3\left(\tfrac{\theta}{2}\right) + \mathrm{Ls}_3\left(\pi-\tfrac{\theta}{2}\right) - \mathrm{Ls}_3(\pi)\right] - \frac{1}{2}I\left(\tfrac{\theta}{2}\right) \, , \quad (32)$$

where we have used the short-hand notation

$$l_\theta = \ln\left(2\sin\tfrac{\theta}{2}\right) \, , \quad L_\theta = \ln\left(2\cos\tfrac{\theta}{2}\right) \, ,$$

the integral $I(\theta)$ is defined as

$$I(\theta) = \int_0^\theta d\phi\phi\left[l_\phi\tan\tfrac{\phi}{2} + L_\phi\cot\tfrac{\phi}{2}\right] \, , \quad (33)$$

and

$$\mathrm{Ls}_j^{(k)}(\theta) = -\int_0^\theta \mathrm{d}\phi\, \phi^k \ln^{j-k-1}\left|2\sin\frac{\phi}{2}\right| \, , \quad (34)$$

is the generalized log-sine function [15]. In particular, $\mathrm{Ls}_j^{(0)}(\theta) = \mathrm{Ls}_j(\theta)$. The integral (33) can be evaluated in terms of the polylogarithms of the complex argument [15]

$$I(\theta) = -\tfrac{3}{2}\pi\zeta_2 - 3\mathrm{Ls}_3(\pi-\theta) - \tfrac{1}{2}\mathrm{Ls}_3(2\theta) + \mathrm{Ls}_3(\theta) + 4\ln 2\left[\mathrm{Ls}_2(\pi-\theta) + \mathrm{Ls}_2(\theta)\right] + 8\Im\mathrm{S}_{1,2}\left(i\tan\tfrac{\theta}{2}\right) + 2\theta\ln\left(\tan\tfrac{\theta}{2}\right)\left[\ln 2 - \ln\left(\cos\tfrac{\theta}{2}\right)\right] \, , \quad (35)$$

where

$$\Im\mathrm{S}_{1,2}\left(i\tan\tfrac{\theta}{2}\right) = \mathrm{Ti}_3\left(\tan\tfrac{\theta}{2}, \cot\tfrac{\theta}{2}\right) + \tfrac{1}{2}\ln\left(\cos\tfrac{\theta}{2}\right)\left[\mathrm{Ls}_2(\pi-\theta) + \mathrm{Ls}_2(\theta) + \theta\ln\left(\tan\tfrac{\theta}{2}\right)\right] - \tfrac{\theta}{8}\mathrm{Li}_2\left(\sin^2\tfrac{\theta}{2}\right) - \tfrac{1}{48}\theta^3 \, , \quad (36)$$

and $\mathrm{Ti}_3(x,a)$ is the inverse tangent integral of two variables. Collecting all expressions we get

$$\rho_{-;-;3}(\theta) = \tfrac{\theta}{4}\mathrm{Li}_2\left(\sin^2\tfrac{\theta}{4}\right) - 4\mathrm{Ti}_3\left(\tan\tfrac{\theta}{4}, \cot\tfrac{\theta}{4}\right) + \mathrm{Ls}_3\left(\pi-\tfrac{\theta}{2}\right) - \mathrm{Ls}_3\left(\tfrac{\theta}{2}\right) + \tfrac{1}{4}\mathrm{Ls}_3(\theta) - \mathrm{Ls}_3(\pi) + \ln\left(\tan\tfrac{\theta}{4}\right)\left[\mathrm{Ls}_2\left(\tfrac{\theta}{2}\right) + \mathrm{Ls}_2\left(\pi-\tfrac{\theta}{2}\right)\right] + \tfrac{\theta^3}{96} - \sin\tfrac{\theta}{2} + \theta\left[\tfrac{1}{4}\ln^2\left(\tan\tfrac{\theta}{4}\right) - \ln\left(2\sin\tfrac{\theta}{4}\right)\ln\left(2\cos\tfrac{\theta}{4}\right)\right] + \tfrac{\theta}{4}L_\theta\left[2\ln\left(\tan\tfrac{\theta}{4}\right) + L_\theta\right] \, . \quad (37)$$

Combining this result with Eqs. (2.61), (2.62) and (2.54) from Ref. [12] we write the one-fold integral representation for the sum $\Sigma^{(1)}_{;-;3;1}(u_\theta)$

$$\Sigma^{(1)}_{-;3;1}(u_\theta) = \frac{1}{8}\tan\tfrac{\theta}{2}\left[2\theta\mathrm{Ls}_3^{(1)}(\theta) - 3\mathrm{Ls}_4^{(2)}(\theta)\right] + \tan\tfrac{\theta}{2}\int_0^\theta d\phi\left(1 + \frac{\rho_{-;-;3}(\phi)}{\sin\frac{\phi}{2}}\right) \, , \quad (38)$$

where $\mathrm{Ls}_3^{(1)}(\theta)$ and $\mathrm{Ls}_4^{(2)}(\theta)$ are expressible in terms of Clausen functions (see Appendix A of Ref. [11].) To obtain a result valid in a different region of variable u_θ $(-4 \le u_\theta \le 0)$, the analytical continuation procedure, firstly described in Ref. [16] (see also [11]) should be applied.

5. Our study [12] allows us to construct some terms of the ε-expansion of the generalized hypergeometric function ${}_{P+1}F_P$ and obtain new analytical results for higher terms of the ε-expansion of the two- and three-loop maser-integrals entering in different packages [17,18].

It was noted, that individual terms of our construction include the log-sine functions of arguments $\theta/2$ and $\pi - \theta/2$ (see Eqs. (37, 38) and Eq. (2.90) in Ref. [12]) which, however, are cancelled in the considered order of ε-expansion of the Feynman diagrams. For single scale massive diagrams, when $\theta = \pi/3$, the arguments of functions are equal to $\pi/6$ and $5\pi/6$, respectively. It opens the question about possible generalization of the "sixth root of unity" basis, introduced in Ref. [19] (for the recent progress see [11,20]).

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